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The Weierstrass semigroup of a pair of Galois Weierstrass points with prime degree on a curve

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Abstract. We describe the Weierstrass semigroup of a Galois Weierstrass point with prime degree and the Weierstrass semigroup of a pair of Galois Weierstrass points with prime degree, where a *Galois Weierstrass point with degree* n means a total ramification point of a cyclic covering of the projective line of degree n.

Keywords: Galois Weierstrass point, Weierstrass semigroup of a point, Weierstrass semigroup of a pair of points.

Mathematical subject classification: Primary: 14H55; Secondary: 14H30, 14H45.

1 Introduction

Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ of H in $\mathbb{N}_0$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H. A numerical semigroup H is called an *n-semigroup* if the least positive integer in H is n. Let C be a complete nonsingular irreducible curve of genus $g \geq 2$ over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. Let $\mathbb{K}(C)$ be the field of rational functions on C. For a point P of C, we set

$$H(P) := \{\alpha \in \mathbb{N}_0 | \text{ there exists } f \in \mathbb{K}(C) \text{ with } (f)_\infty = \alpha P\},$$

which is called the *Weierstrass semigroup of the point* P. We note that $H(P)$ is a numerical semigroup of genus g. An integer n is called the *first non-gap* of P

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if $H(P)$ is an n-semigroup. For distinct points P and Q of C, we set

$$H(P, Q) := \{(\alpha, \beta) \in \mathbb{N}_0 \times \mathbb{N}_0 | \quad \text{there exists} \quad f \in \mathbb{K}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\},$$

which is called the *Weierstrass semigroup of the pair* (P, Q) *of points*. If C is a hyperelliptic curve of genus $g \geq 2$ and P is its point, then the semigroup $H(P)$ is well-known. Moreover, if P and Q are distinct points of the hyperelliptic curve C, Kim [4] determined the semigroup $H(P, Q)$. If C is a curve of genus $g \leq 7$, then every candidate, i.e., every numerical semigroup of genus $g \leq 7$, appears as the Weierstrass semigroup of a point (for the case $g = 4$ see Lax [3], and for the cases $g = 5, 6, 7$ see Komeda [10]). In the case where C is a non-hyperelliptic curve of genus 3, for all distinct points P and Q of C the semigroup $H(P, Q)$ is determined by Kim-Komeda [6]. If P is a point of a curve with first non-gap $a \leq 5$, then every candidate, i.e., every numerical semigroup with first non-gap $a \leq 5$, appears as the Weierstrass semigroup of a point (for the case $a = 3$ see Maclachlan [11] and for the case $a = 4$ (resp. 5) see Komeda [8] (resp. [9])). If P and Q are distinct points whose first non-gaps are 3, then the semigroup $H(P, Q)$ is determined by Kim-Komeda [7].

In Section 2 we give a necessary and sufficient computable condition for a p-semigroup to be the Weierstrass semigroup of a Galois Weierstrass point with degree p where p is a prime number. In Section 3 we determine the Weierstrass semigroup of a pair of Galois Weierstrass points with degree p.

2 The semigroup of a Galois Weierstrass point with prime degree

First we give the notation which we will use in this section. For an n-semigroup H we set $s_i = \text{Min}\{h \in H | h \equiv i \bmod n\}$ for $i = 1, \ldots, n-1$. The set $S(H) = \{n, s_1, \ldots, s_{n-1}\}$ is called the *standard basis for* H. An n-semigroup H is said to be *cyclic* if there is a Galois Weierstrass point P with degree n such that $H(P) = H$. The following result is classical.

Remark 2.1. Any 3-semigroup is cyclic.

Cyclic p-semigroups have the following property:

Remark 2.2. (Morrison-Pinkham [12]). Let p be a prime number. If H is a cyclic p-semigroup, then we have

$$s_i + s_{p-i} = s_j + s_{p-j}, \text{ all } 1 \leq i, j \leq p-1,$$

which are called the *M-P equalities*.

The above condition is a necessary and sufficient condition in the case $p = 5, 7$.

Remark 2.3. If $p = 5$ or 7, then any p-semigroup satisfying the M-P equalities is cyclic (for example, see Morrison-Pinkham [12]).

For an arbitrary prime number p, Theorem 2.1 in Kim-Komeda [5] gives a necessary and sufficient condition for a p-semigroup to be cyclic. Using the theorem we can show that the condition satisfying the M-P equalities is not sufficient for every $p \geq 11$.

Remark 2.4. (Kim-Komeda [5]). If $p \geq 11$, then there exists a non-cyclic p-semigroup satisfying the M-P equalities.

We want to find a strictly additional *computable* condition for a p-semigroup satisfying the M-P equalities to be cyclic. From now on, let p be an odd prime number. We assume that H is a p-semigroup satisfying the M-P equalities. We set

$$S(H) = \{p,\, pa_l + l \ (l = 1, \dots, p-1)\}.$$

We call

$$(\mathrm{I}) \begin{cases} j_1 + \cdots + j_{\frac{p+1}{2}} = a_1 + a_{p-1} + 1 & \\ \displaystyle\sum_{q=1}^{\frac{p+1}{2}} \pi(lq) j_q = pa_l + l & (l = 1, \dots, \frac{p-1}{2}) \end{cases}$$

the *system of linear equations associated to H*, where

$$\pi(x) = x - \left[\frac{x}{p}\right] p$$

for any integer x and [ ] denotes the Gauss symbol. Here $j_1, \dots, j_{\frac{p+1}{2}}$ are the variables. Using Carliz-Olsen [1] we can see that the determinant of the coefficients of (I) is non-zero. Hence (I) has a unique solution. If we can find the solution, we get the necessary and sufficient condition for a p-semigroup satisfying the M-P equalities to be cyclic which will be described in Theorem 2.7.

Proposition 2.5. *Let H be a p-semigroup. Then the following conditions are equivalent.*

i) *H is cyclic.*

ii) $S(H)=\{p\}\cup\left\{\sum_{q=1}^{p-1}\pi(lq)i_q \,\middle|\, l=1,2,\ldots,p-1\right\}$

for some non-negative integers $i_1, i_2, \ldots, i_{p-1}$ with $\sum_{q=1}^{p-1} q i_q \equiv 1$ mod p.

Proof. ii) implies i) by Theorem 2.1 in [5]. We assume that i) holds. Then there is a Galois Weierstrass point P on a curve C such that $H(P)=H$. We may assume that the C is defined by an equation of the form

$$z^p=\prod_{q=1}^{p-1}\prod_{j=1}^{\mu_q}(x-c_{qj})^q \tag{1}$$

where

$$\sum_{q=1}^{p-1} q\mu_q \not\equiv 0 \bmod p$$

and c_{qj}'s are distinct elements of k. Let $f: C \longrightarrow \mathbb{P}^1$ be the morphism corresponding to the inclusion

$$\mathbb{K}(\mathbb{P}^1)=k(x)\subset k(x,z)=\mathbb{K}(C),\ \text{i.e., } f(R)=(1:x(R)).$$

In this case, we may take the point P as $f^{-1}((0:1))=\{P\}$. There exists an integer m with $1\le m\le p-1$ such that

$$m\sum_{q=1}^{p-1} q\mu_q \equiv 1 \bmod p.$$

For any q with $1\le q\le p-1$ we have $mq=n_q p+r_q$ for some integers n_q and r_q with $1\le r_q\le p-1$. Then the m-th power of the equation (1) becomes

$$z^{pm}=\prod_{q=1}^{p-1}\prod_{j=1}^{\mu_q}((x-c_{qj})^{n_q})^p(x-c_{qj})^{r_q}.$$

Hence, if we set

$$Z = \frac{z^m}{\prod_{q=1}^{p-1} \prod_{j=1}^{\mu_q} (x - c_{qj})^{n_q}},$$

we get

$$Z^p = \prod_{q=1}^{p-1} \prod_{j=1}^{\mu_q} (x - c_{qj})^{r_q}$$

with $\sum_{q=1}^{p-1} r_q \mu_q \equiv 1 \bmod p$. Moreover, we have $\mathbb{K}(C) = k(x, z) = k(x, Z)$, because p is prime. By the proof of Theorem 2.1 in Kim-Komeda [5] we have

$$\begin{aligned} S(H(P)) = & \left\{ p, \sum_{q=1}^{p-1} r_q \mu_q, \ldots, \sum_{q=1}^{p-1} \pi(t r_q) \mu_q, \ldots, \sum_{q=1}^{p-1} \pi((p-1) r_q) \mu_q \right\} \\ = & \; S(H). \end{aligned}$$

For any $q = 1, 2, \ldots, p-1$ we set $i_{r_q} = \mu_q$. Then we have

$$\sum_{q=1}^{p-1} \pi(t r_q) \mu_q = \sum_{q=1}^{p-1} \pi(t r_q) i_{r_q} = \sum_{q=1}^{p-1} \pi(t q) i_q.$$

Moreover, we get $\sum_{q=1}^{p-1} q i_q \equiv 1 \bmod p$, because

$$\sum_{q=1}^{p-1} r_q \mu_q = \sum_{q=1}^{p-1} r_q i_{r_q} = \sum_{q=1}^{p-1} q i_q. \qquad \square$$

Proposition 2.6. *Let H be a p-semigroup satisfying the M-P equalities. The semigroup H is cyclic if and only if the system of linear equations*

$$\text{(II)} \begin{cases} i_1 + \cdots + i_{p-1} = a_1 + a_{p-1} + 1 \\ \sum_{q=1}^{p-1} \pi(lq) i_q = p a_l + l \qquad (l = 1, \ldots, \frac{p-1}{2}), \end{cases}$$

has a solution $(i_1, \ldots, i_{p-1}) = (i_1^0, \ldots, i_{p-1}^{(0)})$ consisting of non-negative integers.

Proof. Assume that H is cyclic. By Proposition 2.5 we have

$$S(H) = \{p\} \cup \left\{ \sum_{q=1}^{p-1} \pi(lq) i_q^{(0)} \;\middle|\; l = 1, 2, \ldots, p-1 \right\}$$

for some non-negative integers $i_1^{(0)}, i_2^{(0)}, \ldots, i_{p-1}^{(0)}$ with $\sum_{q=1}^{p-1} q i_q^{(0)} \equiv 1 \bmod p$. Hence, we get

$$\sum_{q=1}^{p-1} \pi(lq) i_q^{(0)} \equiv l \bmod p,$$

which implies that

$$\sum_{q=1}^{p-1} \pi(lq) i_q^{(0)} = pa_l + l \bmod p$$

for all l. Since $q + \pi((p-1)q) = p$ for all q, we have

$$\sum_{q=1}^{p-1} \pi((p-1)q) i_q^{(0)} = \sum_{q=1}^{p-1} (p-q) i_q^{(0)}.$$

Thus, we obtain

$$i_1^{(0)} + \cdots + i_{p-1}^{(0)} = a_1 + a_{p-1} + 1.$$

Therefore, the system (II) has a solution consisting of the non-negative integers $i_1^{(0)}, i_2^{(0)}, \ldots, i_{p-1}^{(0)}$.

Assume that (II) has a solution $(i_1. \ldots, i_{p-1}) = (i_1^{(0)}. \ldots, i_{p-1}^{(0)})$ consisting of non-negative integers. Since H satisfies the M-P equalities and we have

$$\pi(lq) + \pi((p-l)q) = p \text{ for all } q = 1, \ldots, \frac{p-1}{2},$$

we see that

$$\sum_{q=1}^{p-1} \pi(lq) i_q^{(0)} = pa_l + l \;\; (l = 1, \ldots, p-1).$$

Thus, we get

$$S(H) = \{p\} \cup \left\{ \sum_{q=1}^{p-1} \pi(lq) i_q^{(0)} \;\middle|\; l = 1, 2, \ldots, p-1 \right\}.$$

By Proposition 2.5 H must be cyclic. □

Theorem 2.7. *Let H be a p-semigroup satisfying the M-P equalities. Let*

$$(j_1, \ldots, j_{\frac{p+1}{2}}) = (A_1, \ldots, A_{\frac{p+1}{2}})$$

be the unique solution of the system (I) *of linear equations associated to H.*

(1) *If there is $t \in \left\{1, \ldots, \dfrac{p+1}{2}\right\}$ such that A_t is not an integer, then H is non-cyclic.*

(2) *If all A_t's are integers, then the following conditions are equivalent:*

(i) *H is cyclic, i.e., there is a Galois Weierstrass point P with degree p such that $H(P) = H$.*

(ii) $\displaystyle\sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p-1}{2}} \geq 0$ *and* $\displaystyle\sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p+1}{2}} \geq 0$ *where*

$$\mathcal{R}_H := \left\{ r \in \left\{1, \ldots, \frac{p-3}{2}\right\} \;\middle|\; A_r < 0 \right\}.$$

Proof. (1) Consider the system of linear equations

$$\text{(II)} \begin{cases} i_1 + \cdots + i_{p-1} = a_1 + a_{p-1} + 1 \\ \displaystyle\sum_{q=1}^{p-1} \pi(lq) i_q = pa_l + l \qquad\qquad (l = 1, \ldots, \tfrac{p-1}{2}), \end{cases}$$

where $S(H) = \{p, pa_l + l \ (l = 1, \ldots, p-1)\}$. By the assumption we get the solutions of (II)

$$\begin{cases} i_1 = A_1 + i_{p-1} \\ i_2 = A_2 + i_{p-2} \\ \cdots\cdots\cdots\cdots \\ i_{\frac{p-3}{2}} = A_{\frac{p-3}{2}} + i_{\frac{p+3}{2}} \\ i_{\frac{p-1}{2}} = A_{\frac{p-1}{2}} - i_{\frac{p+3}{2}} - \cdots - i_{p-2} - i_{p-1} \\ i_{\frac{p+1}{2}} = A_{\frac{p+1}{2}} - i_{\frac{p+3}{2}} - \cdots - i_{p-2} - i_{p-1}. \end{cases}$$

Assume that there exists $t \in \left\{1, \dots, \dfrac{p+1}{2}\right\}$ such that A_t is not an integer. If H were cyclic, then some solution $(i_1, \dots, i_{p-1})$ must consist of integers by Proposition 2.6. But by the expression of the solutions, i_t is not an integer. This is a contradiction.

(2) Assume that all A_t's are integers. First we prove that (i) implies (ii). Assume that we had

$$\sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p-1}{2}} < 0 \quad \text{or} \quad \sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p+1}{2}} < 0.$$

Since H is cyclic, we get a solution $(i_1, \dots, i_{p-1})$ of (II) consisting of non-negative integers. For any $r \in \left\{1, \dots, \dfrac{p-3}{2}\right\}$ we have $i_r = A_r + i_{p-r} \geq 0$, which implies that $i_{p-r} \geq -A_r$. If $\sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p-1}{2}} < 0$, then we get

$$\begin{aligned} 0 \leq i_{\frac{p-1}{2}} &= A_{\frac{p-1}{2}} - i_{\frac{p+3}{2}} - \cdots - i_{p-2} - i_{p-1} \\ &\leq A_{\frac{p-1}{2}} - \sum_{r \in \mathcal{R}_H} i_{p-r} \leq A_{\frac{p-1}{2}} + \sum_{r \in \mathcal{R}_H} A_r < 0. \end{aligned}$$

This is a contradiction. If $\sum_{r \in \mathcal{R}_H} A_r + A_{\frac{p+1}{2}} < 0$, the same proof works well.

Next we prove that (ii) implies (i). Let

$$s \in \left\{1, \dots, \frac{p-3}{2}\right\}$$

such that $s \notin \mathcal{R}_H$. We set $i_{p-s} = 0$, which implies that $i_s = A_s + i_{p-s} = A_s \geq 0$. Let $r \in \mathcal{R}_H$. We set $i_{p-r} = -A_r > 0$. Then $i_r = A_r + i_{p-r} = A_r - A_r = 0$. Moreover, we have

$$\begin{aligned} i_{\frac{p-1}{2}} &= A_{\frac{p-1}{2}} - i_{\frac{p+3}{2}} - \cdots - i_{p-2} - i_{p-1} \\ &= A_{\frac{p-1}{2}} - \sum_{r \in \mathcal{R}_H} i_{p-r} \\ &= A_{\frac{p-1}{2}} + \sum_{r \in \mathcal{R}_H} A_r \geq 0. \end{aligned}$$

Similarly we get $i_{\frac{p+1}{2}} = A_{\frac{p+1}{2}} + \sum_{r \in \mathcal{R}_H} A_r \geq 0$. Hence we get $i_q \geq 0$ for all $q = 1, \dots, p-1$, which implies that H is cyclic. □

Using Theorem 2.7 we can give an example of a cyclic (resp. non-cyclic) semigroup satisfying the M-P equalities.

Example 2.8. Let H be the 11-semigroup with

$$S(H) = \{11, 23, 24, 25, 26, 27, 39, 40, 41, 42, 43\}$$

$$(\text{resp. } \{11, 12, 16, 18, 19, 20, 24, 25, 26, 28, 32\}).$$

Then H satisfies the M-P equalities. Moreover,

$$\begin{cases} j_1 + j_2 + j_3 + j_4 + j_5 + j_6 = 6 \text{ (resp. 4)} \\ j_1 + 2j_2 + 3j_3 + 4j_4 + 5j_5 + 6j_6 = 23 \text{ (resp. 12)} \\ 2j_1 + 4j_2 + 6j_3 + 8j_4 + 10j_5 + j_6 = 24 \\ 3j_1 + 6j_2 + 9j_3 + j_4 + 4j_5 + 7j_6 = 25 \\ 4j_1 + 8j_2 + j_3 + 5j_4 + 9j_5 + 2j_6 = 26 \\ 5j_1 + 10j_2 + 4j_3 + 9j_4 + 3j_5 + 8j_6 = 27 \text{ (resp. 16)} \end{cases}$$

is the system (I) of linear equations associated to H. The unique solution is $(3, -1, 0, 0, 2, 2)$ (resp. $(1, 1, 1, -1, 2, 0)$), which implies that $\mathcal{R}_H = \{2\}$ (resp. $\{4\}$). Hence we get

$$-1 + 2 \geq 0 \quad \text{and } -1 + 2 \geq 0 \text{ (resp. } -1 + 2 \geq 0 \quad \text{and } -1 + 0 < 0),$$

which implies that H is cyclic (resp. non-cyclic) by Theorem 2.7 (2).

3 The semigroup of a pair of Galois Weierstrass points with prime degree

Throughout this section let C be a curve of genus g. We determine the Weierstrass semigroup at a pair of Galois Weierstrass points P, Q with prime degree. First we review the properties of the semigroup $H(P, Q)$.

Remark 3.1. (Kim [4] and Homma [2]). Let P and Q be distinct points of C. Then we have the following:

i) For each $l \in G(P) = \mathbb{N}_0 \backslash H(P)$, the integer $\mathrm{Min}\{\beta \mid (l, \beta) \in H(P, Q)\}$ must be equal to some element in $G(Q) = \mathbb{N}_0 \backslash H(Q)$, say $\sigma(l)$, and this correspondence σ gives a bijection between the sets $G(P)$ and $G(Q)$.

ii) The semigroup $H(P, Q)$ is completely determined by the bijective correspondence σ, i.e.,

$$G(P,Q) = \bigcup_{l \in G(P)} \Big(\{(l,\beta) | \beta = 0,1,\ldots,\sigma(l)-1\} \cup \{(\alpha,\sigma(l)) | \alpha = 0,1,\ldots,l-1\} \Big)$$

where we set $G(P, Q) = (\mathbb{N}_0 \times \mathbb{N}_0) \backslash H(P, Q)$. Thus, it suffices to determine the graph $\Gamma(P, Q)$ of σ, i.e.,

$$\Gamma(P,Q) = \{(l,\sigma(l)) \mid l \in G(P)\},$$

for describing the semigroup $H(P, Q)$. We call $\Gamma(P, Q)$ the *generating set* for $H(P, Q)$.

Remark 3.2. We can describe the semigroup of a pair of points whose first non-gaps a are 2 (resp. 3) using the generating set (see Kim [4] (resp. Kim-Komeda [7]) for $a = 2$ (resp. 3)).

Let P be a Galois Weierstrass point of degree p on a curve C. By the proof of Proposition 2.5 the curve C can be defined by an equation of the form

$$z^p = \prod_{q=1}^{p-1} \prod_{j=1}^{i_q} (x - c_{qj})^q \tag{2}$$

where

$$\sum_{q=1}^{p-1} q i_q \equiv 1 \bmod p$$

and c_{qj}'s are distinct elements of k. Let $f : C \longrightarrow \mathbb{P}^1$ be the morphism corresponding to the inclusion

$$\mathbb{K}(\mathbb{P}^1) = k(x) \subset k(x,z) = \mathbb{K}(C), \text{ i.e., } f(R) = (1 : x(R)).$$

In this case, we may take the point P as $f^{-1}((0:1)) = \{P\}$. By Theorem 2.1 in Kim-Komeda [5] we have

$$S(H(P)) = \left\{ p, \sum_{q=1}^{p-1} q i_q, \ldots, \sum_{q=1}^{p-1} \pi(tq) i_q, \ldots, \sum_{q=1}^{p-1} \pi((p-1)q) i_q \right\}.$$

Using the above curve C and its point P we get our main theorem.

Theorem 3.3. i) *Let P and Q be distinct Galois Weierstrass points with degree p on a curve C of genus g. Assume that $g > (p-1)^2$. Then there exist non-negative integers $i_1, \dots, i_{p-1}$ with $\sum_{q=1}^{p-1} q i_q \equiv 1 \bmod p$ and an integer s with $1 \le s \le p-1$ satisfying $i_s > 0$ such that*

$$S(H(P)) = \left\{ p, \sum_{q=1}^{p-1} q i_q, \dots, \sum_{q=1}^{p-1} \pi(tq) i_q, \dots, \sum_{q=1}^{p-1} \pi((p-1)q) i_q \right\},$$

$$S(H(Q)) = \Bigg\{ p, \sum_{q=1}^{p-1} q i_q + p - 1 - s, \dots, \sum_{q=1}^{p-1} \pi(tq) i_q + p - t - \pi(ts), \dots,$$
$$\sum_{q=1}^{p-1} \pi((p-1)q) i_q + 1 - \pi((p-1)s) \Bigg\}$$

and

$$\Gamma(P, Q) = \Bigg\{ \left(\sum_{q=1}^{p-1} \pi(mq) i_q - lp, lp - \pi(ms) \right) \Bigg|$$
$$1 \le l \le \left[\frac{\sum_{q=1}^{p-1} \pi(mq) i_q}{p} \right], 1 \le m \le p-1 \Bigg\}.$$

ii) *Conversely, let $i_1, \dots, i_{p-1}$ be non-negative integers such that*

$$\sum_{q=1}^{p-1} q i_q \equiv 1 \bmod p.$$

Take an integer s with $i_s > 0$. Then we can construct a pair (P, Q) of Galois Weierstrass points with degree p such that $S(H(P))$, $S(H(Q))$ and $\Gamma(P, Q)$ are as in i)*.*

Proof. i) Let C be the curve with the equation (2). We set $f^{-1}((1 : c_{st})) = \{P_{st}\}$. Since the genus of C is larger than $(p-1)^2$, we have $Q = P_{st}$ for some s and t. We transform the variable x by $X = \dfrac{1}{x - c_{st}}$. Then the equation (2) becomes

$$\frac{1}{c} z^p X^{\sum_{q=1}^{p-1} q i_q} = \left(\prod_{q=1,\neq s}^{p-1} \prod_{j=1}^{i_q} (X - c'_{qj})^q \right) \prod_{j=1,\neq t}^{i_s} (X - c'_{sj})^s$$

where $c'_{qj} = \dfrac{1}{c_{qj} - c_{st}}$ and c is some constant. Then we get

$$Z^p = X^{p-1}\left(\prod_{q=1,\neq s}^{p-1}\prod_{j=1}^{i_q}(X - c'_{qj})^q\right)\prod_{j=1,\neq t}^{i_s}(X - c'_{sj})^s,$$

where we set $Z = c^{-\frac{1}{p}}X^{\frac{u}{p}}z$ and $u = \sum_{q=1}^{p-1} qi_q + p - 1$. If $s = p-1$, then we get

$$Z^p = \left(\prod_{q=1}^{p-2}\prod_{j=1}^{i_q}(X - c'_{qj})^q\right)\left(X^{p-1}\prod_{j=1,\neq t}^{i_{p-1}}(X - c'_{p-1j})^{p-1}\right).$$

If $s \neq p-1$, then we obtain

$$Z^p = \left(\prod_{q=1,\neq s}^{p-2}\prod_{j=1}^{i_q}(X - c'_{qj})^q\right)\left(\prod_{j=1,\neq t}^{i_s}(X - c'_{sj})^s\right)\left(X^{p-1}\prod_{j=1}^{i_{p-1}}(X - c'_{p-1j})^{p-1}\right).$$

If $s = p-1$, then

$$S(H(Q)) = S(H(P)) = \{p\} \cup \left\{\sum_{q=1}^{p-1}\pi(tq)i_q \,|\, t = 1, 2, \dots, p-1\right\}$$

If $s \neq p-1$, then by Theorem 2.1 in Kim-Komeda [5] we have

$$\begin{aligned} S(H(Q)) = \{p\} \cup \Bigg\{ \sum_{q=1,\neq s}^{p-2} \pi(tq)i_q + \pi(ts)(i_s - 1) + \pi(t(p-1))(i_{p-1} + 1) \,|\, \\ t = 1, 2, \dots, p-1\Bigg\} \\ = \{p\} \cup \left\{\sum_{q=1}^{p-1}\pi(tq)i_q + \pi(t(p-1)) - \pi(ts) \,|\, t = 1, 2, \dots, p-1\right\} \\ = \{p\} \cup \left\{\sum_{q=1}^{p-1}\pi(tq)i_q + p - t - \pi(ts) \,|\, t = 1, 2, \dots, p-1\right\}. \end{aligned}$$

For any positive integer l and any $m = 1, 2, \dots, p-1$, consider the divisor

$$\left(\frac{z^m}{(x - c_{st})^l \prod_{q=1}^{p-1}\prod_{j=1}^{i_q}(x - c_{qj})^{[\frac{mq}{p}]}}\right)$$

$$= m\left(\sum_{q=1}^{p-1}\sum_{j=1}^{i_q} qP_{qj} - \sum_{q=1}^{p-1} qi_q P\right) - l(pP_{st} - pP)$$

$$-\left(\sum_{q=1}^{p-1}\sum_{j=1}^{i_q}\left[\frac{mq}{p}\right] pP_{qj} - \sum_{q=1}^{p-1}\left[\frac{mq}{p}\right] pi_q P\right)$$

$$= \sum_{q=1,\neq s}^{p-1}\sum_{j=1}^{i_q}\left(mq - \left[\frac{mq}{p}\right]p\right)P_{qj} + \sum_{j=1,\neq t}^{i_s}\left(ms - \left[\frac{ms}{p}\right]p\right)P_{sj}$$

$$-\left(lp - ms + \left[\frac{ms}{p}\right]p\right)P_{st} - \left(m\sum_{q=1}^{p-1} qi_q - \sum_{q=1}^{p-1}\left[\frac{mq}{p}\right]pi_q - lp\right)P$$

$$= \sum_{q=1,\neq s}^{p-1}\sum_{j=1}^{i_q}\pi(mq)P_{qj} + \sum_{j=1,\neq t}^{i_s}\pi(ms)P_{sj}$$

$$-(lp - \pi(ms))P_{st} - \left(\sum_{q=1}^{p-1}\pi(mq)i_q - lp\right)P.$$

We note that $lp - \pi(ms) > 0$. Moreover, if $l \leq \left[\frac{\sum_{q=1}^{p-1}\pi(mq)i_q}{p}\right]$, then

$$\sum_{q=1}^{p-1}\pi(mq)i_q - lp > 0.$$

Hence, for $1 \leq m \leq p-1$ and $1 \leq l \leq \left[\frac{\sum_{q=1}^{p-1}\pi(mq)i_q}{p}\right]$ we get

$$\left(\sum_{q=1}^{p-1}\pi(mq)i_q - lp, lp - \pi(ms)\right) \in H(P, Q).$$

By Lemma 2 in Homma [2] we get the result.

ii) Using the integers $i_1, \ldots, i_{p-1}$ we construct the curve C with the equation (2) and its point P. Let us take P_{s1} as Q where $f^{-1}((1 : c_{s1})) = \{P_{s1}\}$. Then we get the desired result. □

We give an example of the semigroup of a pair of Galois Weierstrass points such that we can take only one s as in the above theorem.

Example 3.4. Let H be the 11-semigroup with

$$S(H) = \{11, 23, 46, 69, 92, 115, 138, 161, 184, 207, 230\}.$$

It satisfies the M-P equalities. The solution $(A_1, \dots, A_6)$ of the system (I) associated to H is $(23, 0, 0, 0, 0, 0)$, which implies that $\mathcal{R}_H = \emptyset$. Hence H is cyclic. The solutions of the system (II) in the proof of Theorem 2.7 (1) are

$$i_1 = 23 + i_{10}, \quad i_2 = i_9, \quad i_3 = i_8, \quad i_4 = i_7,$$

$$i_5 = -i_7 - i_8 - i_9 - i_{10}, \quad i_6 = -i_7 - i_8 - i_9 - i_{10},$$

where i_7, i_8, i_9 and i_{10} are arbitrary. If $i_1, i_2, \dots, i_{10}$ are non-negative, then we must have $i_q = 0$ for all $q = 2, 3, \dots, 10$. Thus, $(23, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ is only one solution of (II) consisting of non-negative integers, which means that $i_s > 0$ implies $s = 1$. By Theorem 3.3 ii) we can construct Galois Weierstrass points P and Q such that

$$H(P) = H, \ S(H(Q)) = \{11, 32, 53, 74, 95, 116, 137, 158, 179, 200, 221\}$$

and

$$\Gamma(P, Q) = \{(23m - 11l, 11l - m) \mid 1 \leq l \leq 2m, 1 \leq m \leq 10\}.$$

In fact, let C be the curve defined by

$$z^{11} = \prod_{j=1}^{23}(x - c_{1j}) \quad \text{and} \quad f : C \longrightarrow \mathbb{P}^1$$

the morphism corresponding to the inclusion $k(x) \subset k(x, z)$. Set $\{P\} = f^{-1}((0\colon 1))$ and $\{Q\} = f^{-1}((1\colon c_{11}))$. We get the desired one.

For the following cyclic 11-semigroup H we may take any s with $1 \leq s \leq 10$ as in the above theorem.

Example 3.5. Let H be the 11-semigroup with

$$S(H) = \{11, 89, 90, 146, 92, 93, 149, 150, 96, 152, 153\}.$$

It satisfies the M-P equalities. The solution $(A_1, \dots, A_6)$ of the system (I) associated to H is $(6, 0, 5, -5, 8, 8)$, which implies that $\mathcal{R}_H = \{4\}$. Since we have

$$A_4 + A_5 = -5 + 8 \geq 0 \text{ and } A_4 + A_6 = -5 + 8 \geq 0,$$

we see that H is cyclic by Theorem 2.7 (2). The solutions of the system (II) in the proof of Theorem 2.7 (1) are

$$i_1 = 6 + i_{10}, \quad i_2 = i_9, \quad i_3 = 5 + i_8, \quad i_4 = -5 + i_7,$$

$$i_5 = 8 - i_7 - i_8 - i_9 - i_{10}, \quad i_6 = 8 - i_7 - i_8 - i_9 - i_{10},$$

where i_7, i_8, i_9 and i_{10} are arbitrary. For example, (6, 1, 5, 1, 1, 1, 6, 0, 1, 0) and (7, 0, 6, 0, 1, 1, 5, 1, 0, 1) are solutions of (II) consisting of non-negative integers. Therefore for any s we have a solution $(i_1, \dots, i_{10})$ of (II) consisting of non-negative integers such that $i_s > 0$. In this example we set $s = 2$. Namely, let $(i_1, \dots, i_{10}) = (6, 1, 5, 1, 1, 1, 6, 0, 1, 0)$. Then by Theorem 3.3 ii) we can construct Galois Weierstrass points P and Q such that

$$H(P) = H,\ S(H(Q)) = \{11, 97, 95, 148, 91, 89, 153, 151, 94, 147, 145\}$$

and

$$\Gamma(P, Q) =$$
$$\{(89 - 11l, 11l - 2) \mid l = 1, \dots, 8\} \cup \{(90 - 11l, 11l - 4) \mid l = 1, \dots, 8\} \cup$$
$$\{(146 - 11l, 11l - 6) \mid l = 1, \dots, 13\} \cup \cdots$$
$$\cdots \cup \{(153 - 11l, 11l - 9) \mid l = 1, \dots, 13\}.$$

In fact, let C be the curve defined by

$$z^{11} = \prod_{j=1}^{6}(x - c_{1j}) \cdot (x - c_{21})^2 \cdot \prod_{j=1}^{5}(x - c_{3j})^3 \cdot (x - c_{41})^4$$

$$\cdot (x - c_{51})^5 \cdot (x - c_{61})^6 \prod_{j=1}^{6}(x - c_{7j})^7 \cdot (x - c_{91})^9.$$

We denote by $f : C \longrightarrow \mathbb{P}^1$ the morphism corresponding to the inclusion $k(x) \subset k(x, z)$. Set $\{P\} = f^{-1}((0 : 1))$ and $\{Q\} = f^{-1}((1 : c_{21}))$. Then we get the desired one.

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